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INVERTIBILITY OF AFFINE NONLINEAR CONTROL SYSTEMS: A GEOMETRIC APPROACH

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Invertibility of affine nonlinear control systems: a geometric approach*)

by

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ABSTRACT

The paper deals with the invertibility of multivariable nonlinear control systems. By using the recently developed theory on controlled invariant and controllability distributions necessary and sufficient conditions for invertibility are derived.

KEY WORDS & PHRASES: *nonlinear control systems, invertibility, controlled invariance, controllability distributions*

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*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

Consider the affine nonlinear control system

$$(1.1)\qquad \begin{cases} \dot{x}(t) = A(x(t)) + \sum_{i=1}^{m} B_i(x(t))u_i(t), & x(0) = x_0 \in M \\ y(t) = C(x(t)) \end{cases}$$

where x are local coordinates of an analytic n-dimensional manifold M, $A, B_1, \ldots, B_m$ are analytic vector fields on M and the inputfunction $u(t) = (u_1(t), \ldots, u_m(t))$ belongs to $\mathcal{U}$, the class of analytic functions from $[0,\infty)$ into $\mathbb{R}^m$. Furthermore C is the analytic output map from the state space M into the analytic p-dimensional output-manifold N. We will assume that C is a surjective submersion. The system (1.1) is said to be invertible if the corresponding input-output map is injective. A refined notion is given by: the system (1.1) is invertible at $x_0 \in M$, if whenever u and $\hat{u}$ are distinct admissible controls, then the corresponding outputfunctions $y(\cdot,u,x_0)$ and $y(\cdot,\hat{u},x_0)$ are different. The system is strongly invertible at x_0 if the system is invertible for each x in V for some neighborhood V of x_0 and the system is said to be strongly invertible if there exists an open and dense submanifold M_0 of M such that for all $x_0 \in M_0$ the system is strongly invertible at x_0. The above definitions come from HIRSCHORN ([2,3,4]), who firstly studied nonlinear invertibility. For multivariable linear systems there are several different ways to characterize invertibility. Shortly said Hirschorn's approach is the nonlinear version of that of SILVERMAN ([12]). In that way one constructs a left-inverse system for the original system (see also SINGH [13]). The approach we present here is completely different from that of Hirschorn. Based on the recent developed theory on nonlinear controlled invariance, see e.g. HIRSCHORN ([5]), ISIDORI et al. ([6]), NIJMEIJER & van der SCHAFT ([10]), we will set up a geometric theory for nonlinear invertibility. The basic objects we need here are the so-called controllability distributions, a special class of controlled invariant distributions, introduced in NIJMEIJER ([8], see also KRENER & ISIDORI [7]). In this way we obtain a result which seems to be an improvement of REBHUHN [11].

The outline of the paper is as follows. In section 2 we will derive a geometric condition for strong invertibility of multivariable linear systems. In section 3 we study single input nonlinear systems, while in section 4 we deal with multivariable nonlinear systems.

2. STRONG INVERTIBILITY OF MULTIVARIABLE LINEAR SYSTEMS

Consider the linear system

$$(2.1)\qquad \begin{cases} \dot{x} = Ax + Bu, \quad x(0) = x_0 \\ \\ y = Cx \end{cases}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ and A,B and C matrices of appropriate dimensions. Furthermore we assume that the matrix B has full rank (otherwise the system (2.1) is never invertible). Invertibility of (2.1) at $x_0 \in \mathbb{R}^n$ can easily be expressed in geometric terms. Recall that a subspace $\mathcal{R} \subset \mathbb{R}^n$ is called a *controllability subspace* of the system (2.1) if there exists a linear map $F:\mathbb{R}^n \to \mathbb{R}^m$ such that $\mathcal{R} = \langle A+BF | \mathcal{B}\cap\mathcal{R}\rangle := \mathcal{B} \cap \mathcal{R} + (A+BF)\mathcal{B} \cap \mathcal{R} + \ldots + (A+BF)^{n-1}\mathcal{B} \cap \mathcal{R}$ where $\mathcal{B} := \operatorname{Im} B$ (see WONHAM [16]). The maximal controllability subspace contained in Ker C - which does exist, cf. [1] - will be denoted by $\mathcal{R}^*$. Then we have

<u>THEOREM 2.1</u>. *The system* (2.1) *is strongly invertible at* x_0 *if and only if* $\mathcal{R}^* = \underline{0}$.

<u>PROOF</u>. ($\Rightarrow$) Suppose that $\mathcal{R}^* \neq \underline{0}$. Then there exists a feedback matrix F such that $\mathcal{R}^* = \langle A+BF | \mathcal{B}\cap\mathcal{R}^*\rangle$. Clearly, cf. [16], there exists an (m,m)-matrix $G \neq 0$ such that $\operatorname{Im}(BG) = \mathcal{B} \cap \mathcal{R}^*$. Now consider the 'subsystem' ([16,8]) of (2.1):

$$(2.2)\qquad \begin{cases} \dot{x} = (A+BF)x + BGv, \quad x(0) = x_0 \\ \\ y = Cx \end{cases}$$

Clearly all inputs $u = Fx + Gv$ in the system (2.1) give rise to the same

output function; i.e. the system (2.1) is not invertible at x_0.
($\Leftarrow$) Suppose that the system (2.1) is not strontly invertible at x_0. Then there exist input functions $u_1(.)$ and $u_2(.)$ such that their corresponding output functions coincide, i.e. for all $t \geq 0$

$$Ce^{At}x_0 + C\int_0^t e^{A(t-\sigma)}Bu_1(\sigma)d\sigma = Ce^{At}x_0 + C\int_0^t e^{A(t-\sigma)}Bu_2(\sigma)d\sigma$$

$$\Rightarrow \int_0^t e^{A(t-\sigma)}B(u_1-u_2)(\sigma)d\sigma \in \mathrm{Ker}C, \ \forall t \geq 0.$$

Now if we define $\mathcal{V} = \underset{t\geq 0}{\mathrm{Span}}\left[\int_0^t e^{A(t-\sigma)}B(u_1-u_2)(\sigma)d\sigma\right]$,

then we see that this linear subspace $\mathcal{V}$ is controlled invariant for the system (2.1). Furthermore it is easy to see that $\mathcal{V} \cap \mathcal{B} \neq 0$, which implies that $\mathcal{R}^* \neq 0$. □

<u>REMARK</u>. The linearity of the system (2.1) implies that if the system is strongly invertible at a point x_0, then it is strongly invertible everywhere.

Throughout the paper we are especially interested in a special class of controllability subspaces, namely we will mainly deal with those controllability subspaces $\mathcal{R}$ for which $\mathcal{R} = \langle A+BF,b\rangle$ for an n-vector $b \in \mathcal{B}$ and a feedback matrix F. We will call such an $\mathcal{R}$ a single-input controllability subspace. By using Heymann's lemma (see e.g. [16]), it is easily shown that every controllability subspace $\mathcal{R}$ can be written as a single input controllability subspace. We then have

<u>COROLLARY 2.2</u>. *The system* (2.1) *is strongly invertible* (*at* x_0) *if and only if there is no single-input controllability subspace contained in* Ker C.

3. STRONG INVERTIBILITY OF SINGLE-INPUT NONLINEAR SYSTEMS

Now we consider the affine nonlinear system

$$(3.1) \qquad \begin{cases} \dot{x} = A(x) + B(x)u \\ y = C(x) \end{cases}$$

where $x \in M$, $y \in N$, M and N,A,B and C are as in the introduction. Necessary and sufficient conditions for strong invertibility (at a point x_0) for (3.1) can be found in [2]. In what follows we will frequently use those results. As in section 2 we have to investigate controllability subspaces for (3.1). Clearly, in general for nonlinear systems such subspaces do not exist. The suitable generalization we need is given in [8], which uses to a large extent the notion of *accessibility* of SUSSMANN & JURDJEVIC ([14]). Recall the following definition (cf. [8])

DEFINITION 3.1. The *accessibility distribution* D of the system (3.1) is given by

$$D := \text{involutive closure of } \{ad_A^j B,\ j = 0,1,2,\ldots\}.$$

REMARKS (i). As usual $ad_A^j B$ is defined as follows $ad_A^0 B = B$, $ad_A^{j+1} B = [A, ad_A^j B]$, $j = 0,1,2,\ldots$.

(ii) The distribution D is a *regular* controllability distribution for the system (3.1), see [8]. As in the linear case - for a single-input linear system there exists only one controllability subspace - D is the only controllability distribution for the system (3.1). This is almost trivial, while D is invariant under state-feedback (cf. [8]).

(iii) In [8] we only considered controllability distributions of fixed dimension. Without this assumption the above definition is the obvious extension. It is well known that for an analytic system the accessibility distribution has fixed dimension on an open and dense submanifold M' of M (cf. [14]).

The output map $C:M \to N$, which is a surjective submersion, induces another involutive distribution of fixed dimension on M, namely Ker C_*. Obviously if we consider the two distributions D and Ker C_*, then we have two possibilities. Or the distribution D is contained in Ker C_*, or there exists an open and dense submanifold M_0 of M such that on M_0 D is not contained in Ker C_*. We now obtain:

THEOREM 3.1. *The system (3.1) is strongly invertible if and only if there exists an open and dense submanifold* M_0 *of* M *such that on* M_0 *the distribu-*

D *is not contained in* Ker C_*.

And as a local result we have

THEOREM 3.2. *The system (3.1) is strongly invertible at* x_0 *if and only if there exists a neighborhood* $U(x_0)$ *such that on* $U(x_0)$ *the distribution* D *is not contained in* Ker C_*.

PROOF (of Theorem 3.1). (⇒) Trivial, while if $D \subset \text{Ker } C_*$ the output function is independent of the input, for each initial state x_0.
(⇐) This follows from [2]. A more direct argument, not using a left-inverse for the system (3.1), goes as follows. Suppose that $u_1, u_2 \in \mathcal{U}$ are two different input functions. From the analyticity it follows that for some $\bar{\beta} \in \mathbb{N}$ we have $u_1^{(\bar{\beta})}(0) \neq u_2^{(\bar{\beta})}(0)$ $\left(u^{(\beta)}(0) := \frac{d^\beta u}{dt^\beta}(0)\right)$. For each point x_0 in M_0 the subspace $D(x_0) \subset T_{x_0}M_0$ is generated by the vectors $\text{ad}_A^k B(x_0)$, $k \in \mathbb{N}$ (precisely $D(x_0)$ = involutive closure of $\{\text{ad}_A^k B(x_0),\ k \in \mathbb{N}\}$). Choose local coordinates around x_0 and $C(x_0)$ and let c_i be the i-th component of the output function $(i=1,\dots,p)$. Now $D(x_0)$ is not contained in $(\text{Ker } C_*)(x_0)$, therefore there exists an $\alpha(x_0) \in \mathbb{N}\backslash\{0\}$ such that $(\text{ad}_A^{k-1} B)c_i(x_0) = 0$, $k = 1,\dots,\alpha(x_0)$, $i = 1,\dots,p$ and for some i $\text{ad}_A^{\alpha(x_0)} Bc_{i_0}(x_0) \neq 0$. By using the analyticity of the system (3.1) we see that the number $\alpha(x)$, $x \in M_0$ is a constant $\bar{\alpha}$ on an open and dense submanifold M_0' of M_0, and therefore also on an open and dense submanifold of M. Usually this $\bar{\alpha}$ is called the *relative order* of the system (c.f. [2,3]). Suppose that $x_0 \in M_0'$. We have to show that the output functions which correspond to the input functions u_1 and u_2 are different. Without loss of generality we may assume that $p = 1$ and we write $y_1(t)$ (respectively $y_2(t)$) as the output function of the system (3.1) with initial state x_0 and input function $u_1(t)$ (respectively $u_2(t)$). Then we have

$$(3.2)\qquad y_1(0) - y_2(0) = C(x_0) - C(x_0) = 0$$

$$(3.3)\qquad \begin{aligned}\dot{y}_1(0) - \dot{y}_2(0) &= (L_AC)(x_0) + (L_BC)(x_0).u_1(0) - (L_AC)(x_0) - (L_BC)(x_0)u_2(0)\\ &= (L_BC)(x_0).[u_1(0) - u_2(0)]\end{aligned}$$

Now the right-hand side of (3.3) is different from zero if $\alpha = 1$ and $\beta = 0$. In all other cases we see that $\dot{y}_1(0) - \dot{y}_2(0)$ vanishes. In the next step we obtain

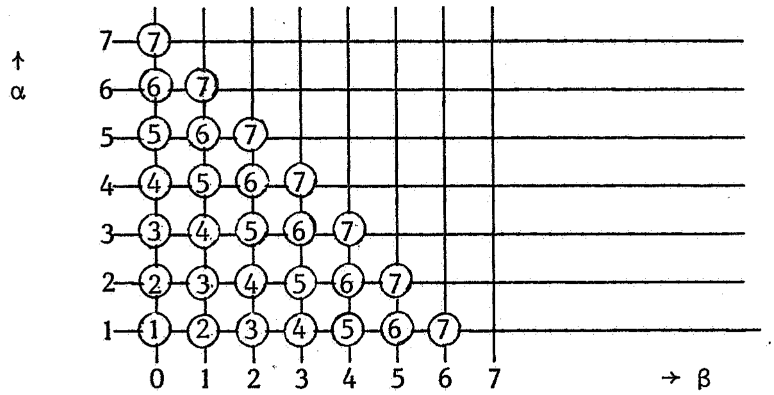
$$
\begin{aligned}
\ddot{y}_1(0) - \ddot{y}_2(0) &= (L_AL_AC)(x_0)+(L_AL_BC)(x_0)u_1(0)+(L_BL_AC)(x_0)u_1(0)\\
&\quad (L_BL_BC)(x_0)u_1^2(0)+(L_BC)(x_0)\dot{u}_1(0)-(L_AL_AC)(x_0)\\
&\quad - (L_AL_BC)(x_0)u_2(0)-(L_BL_AC)(x_0)u_2(0)-(L_BL_BC)(x_0)u_2^2(0)\\
&\quad - (L_BC)(x_0)\dot{u}_2(0)\\
&= (L_AL_BC)(x_0)[u_1(0)-u_2(0)]+(L_BL_AC)(x_0)[u_1(0)-u_2(0)]\\
&\quad + (L_BL_BC)(x_0)[u_1^2(0)-u_2^2(0)]+(L_BC)(x_0)[\dot{u}_1(0)-\dot{u}_2(0)]
\end{aligned}
\tag{3.4}
$$

Assuming that $(\alpha,\beta) \neq (1,0)$ the right-hand side of (3.4) does not vanish if and only if $\alpha = 1$ and $\beta = 1$ or if $\alpha = 2$ and $\beta = 0$ (note that if $\alpha = 2$ the expressions $(L_AL_BC)(x_0)$ and $(L_BL_BC)(x_0)$ vanish on M_0'). In all other cases $\ddot{y}_1(0) - \ddot{y}_2(0) = 0$.

Similarly we obtain

$$
\begin{aligned}
y_1^{(3)}(0) - y_2^{(3)}(0) &= (L_AL_AL_BC)(x_0)[u_1(0)-u_2(0)]+(L_AL_BL_AC)(x_0)[u_1(0)-\\
&\quad - u_2(0)] +\\
&\quad (L_AL_BL_BC)(x_0)[u_1^2(0)-u_2^2(0)] +\\
&\quad (L_BL_AL_AC)(x_0)[u_1(0)-u_2(0)]+(L_BL_AL_BC)(x_0)[u_1^2(0)-\\
&\quad - u_2^2(0)] +\\
&\quad (L_BL_BL_AC)(x_0)[u_1^2(0)-u_2^2(0)]+(L_BL_BL_BC)(x_0)[u_1^3(0)-\\
&\quad - u_2^3(0)] +\\
&\quad 3(L_BL_BC)(x_0)[\dot{u}_1(0)u_1(0)-\dot{u}_2(0)u_2(0)] +\\
&\quad 2(L_AL_BC)(x_0)[\dot{u}_1(0)-\dot{u}_2(0)]+(L_BL_AC)(x_0)[\dot{u}_1(0)-\\
&\quad - \dot{u}_2(0)] +\\
&\quad (L_BC)(x_0)[\ddot{u}_1(0)-\ddot{u}_2(0)]
\end{aligned}
\tag{3.5}
$$

Excluding the three preceding possibilities $(\alpha,\beta) = (1,0)$, $(\alpha,\beta) = (1,1)$ and $(\alpha,\beta) = (2,0)$ we see that the right-hand side of (3.5) does not vanish if and only if $\alpha = 3$ and $\beta = 0$ or $\alpha = 2$ and $\beta = 1$ or $\alpha = 1$ and $\beta = 2$. In general we obtain a lattice



where the index j at a vertex indicates that $y_1^{(j)}(0) - y_2^{(j)}(0)$ is non-zero. In our case the $(\bar{\alpha}+\bar{\beta})$-th derivative of y_1-y_2 is different from zero at time $t = 0$, i.e. the output functions $y_1(t)$ and $y_2(t)$ are different. □

REMARK. Compare the proof of the above theorem with [2], [11], [9] and also the Fliess' approach, e.g. [1].

The proof of theorem 3.2 is completely similar and we will leave it for the reader.

4. STRONG INVERTIBILITY OF MULTIVARIABLE NONLINEAR SYSTEMS

Consider the affine nonlinear system

$$(4.1)\qquad \begin{cases} \dot{x} = A(x) + \sum_{i=1}^{m} B_i(x)u_i \\ y = C(x) \end{cases}$$

where x, y, A, B_i and C are as in the introduction. For studying invertibility of (4.1) we will assume throughout this section that the vector fields $B_1, \ldots, B_m$ are linearly independent at each point of M. Recall the following definitions of [8] (See also [7]).

DEFINITION 4.1. A(k-dimensional) *subsystem* of the system (4.1) is given by

$$(4.2)\qquad \begin{cases} \dot{x} = \tilde{A}(x) + \sum_{j=1}^{k} \tilde{B}_j(x)v_i \\ y = C(x) \end{cases}$$

where

$$\tilde{A}(x) = A(x) + \sum_{i=1}^{m} B_i(x)\alpha_i(x)$$

$$\tilde{B}_j(x) = \sum_{i=1}^{m} B_i(x)\beta_{ij}(x), \qquad j = 1,\dots,k$$

for analytic functions α_i, β_{ij}, $M \to \mathbb{R}$, $i = 1,\dots,m$, $j = 1,\dots,k$. We will call a k-dimensional subsystem *nontrivial* at x_0 if the rank of the matrix $(\beta_{ij}(x_0))_{i,j}$ is greater than zero. Finally, for a neighborhood V in M a k-dimensional subsystem is V-nontrivial if the rank of the matrix $(\beta_{ij})_{ij}$ is greater than zero on V.

DEFINITION 4.2. A *controllability distribution* D of the system (4.1) is the accessibility distribution of a subsystem (4.2) of the system (4.1), i.e. D is the involutive closure of

$$\{ad^{\ell}_{\tilde{A}}\, \tilde{B}_j,\ j = 1,\dots,k,\ \ell \in \mathbb{N}\}.$$

REMARK. In general a controllability distribution D of (4.1) is not locally controlled invariant for the original system (4.1), but it is locally controlled invariant for the sub-system (4.2). If the distribution D is also controlled invariant for (4.1) it is called a *regular* controllability distribution of (4.1), see [8].

THEOREM 4.3. *The system (4.1) is strongly invertible at* x_0 *if and only if there exists a neighborhood* V *of* x_0 *such that each* V*-nontrivial single-input subsystem (4.2) of (4.1) is strongly invertible at* x_0.

PROOF. (⇒) Suppose that the system is strongly invertible at x_0, i.e. the corresponding input-output map is injective on $\mathcal{U}$. Consider an arbitrary subsystem (4.2) which is nonsingular at x_0. Clearly the set of analytic

input functions of the subsystem (4.2) can be inbedded as a subset $\mathcal{U}_1$ of $\mathcal{U}$. While the input-output map is injective on $\mathcal{U}$ it certainly is injective on $\mathcal{U}_1$, which implies that the subsystem is strongly invertible at x_0.
($\Leftarrow$) Suppose that the system (4.1) is not strongly invertible at x_0. Then either it is not invertible at x_0, or it is not invertible at a point $\bar{x}_0 \in V$. We will only consider the first possibility; otherwise the same arguments can be applied by replacing $\bar{x}_0$ instead of x_0.

So suppose that the system (4.1) is not invertible at x_0. Then there exist two different analytic input functions $u(t) = (u_1(t),\ldots,u_m(t))$ and $\bar{u}(t) = (\bar{u}_1(t),\ldots,\bar{u}_m(t))$ which give rise to the same output function. We will show that there exists a single-input subsystem, which is non-singular at x_0, such that both the trajectories of (4.1) corresponding to the input functions $u(t)$ and $\bar{u}(t)$ are also trajectories of this subsystem. Clearly this gives us a contradiction with the fact that each single-input subsystem is strongly invertible at x_0. Let $x(t)$, respectively $\bar{x}(t)$; denote the solution of (4.1) with input function $u(t)$, respectively $\bar{u}(t)$ and initial state $x(0) = x_0$. We may assume that there exists an $\varepsilon > 0$ such that the map from $[0,\varepsilon)$ into M, defined by $t \to x(t)$ is an injective immersion (otherwise we can take $\bar{x}(t)$). Therefore along this trajectory $T = \{x(t) | 0 \leq t < \varepsilon\}$ we can define a state feedback $\bar{\alpha}: T \to \mathbb{R}^m$ in the following way. Let $x_1 \in T$, then there is an unique $\bar{t} \in [0,\varepsilon)$ such that $x(\bar{t}) = x_1$. Define $\bar{\alpha}_i(x_1) = u_i(\bar{t})$, $i = 1,\ldots,m$. While T is an injectively immersed submanifold of M, we can extend the feedback $(\bar{\alpha}_1,\ldots,\bar{\alpha}_m)$ to a feedback $(\alpha_1,\ldots,\alpha_m)$ on a neighborhood in M of T, i.e. on T we have $\alpha_i = \bar{\alpha}_i$.

Define the vector field $\tilde{A}$ by $\tilde{A}(x) = A(x) + \sum_{i=1}^m B_i(x)\alpha_i(x)$. Note that T is the trajectory of the vector field $\tilde{A}$ through x_0. In the same way we can treat $\bar{x}(t)$. There are two possibilities

(i) There exists an $\bar{\varepsilon} > 0$ such that $\bar{T} = \{\bar{x}(t) | 0 \leq t < \bar{\varepsilon}\}$ is also an injectively immersed submanifold of M. Clearly by choosing ε and $\bar{\varepsilon}$ sufficiently small we can achieve that $T \cap \bar{T} = \{x_0\}$. Furthermore we can construct a vector field $\tilde{B}$ locally, $\tilde{B}(x) = \Sigma_{i=1}^m B_i(x)\beta_i(x)$ and $\bar{T}$ is the trajectory of the vector field $\tilde{A} + \tilde{B}$ through x_0. By choosing $\bar{\varepsilon}$ sufficiently small we can also achieve that $\tilde{B}$ is non-zero on a neighborhood of x_0.

(ii) For all $t > 0$ we have that $\bar{x}(t) = x_0$. The construction of an appropriate feedback function $(\beta_1,\ldots,\beta_m)$ as above now becomes almost trivial, while such a feedback function now is only specified in x_0. Again we find that locally there exists a vector field $\tilde{B}$, such that $\bar{T}$ is the trajectory of $\tilde{A} + \tilde{B}$ through x_0. Now consider the single-input subsystem

$$(4.3) \qquad \begin{cases} \dot{x} = \tilde{A}(x) + \tilde{B}(x)v, \quad x(0) = x_0 \\ y = C(x) \end{cases}$$

which by construction is nonsingular at a neighborhood of x_0. The input functions $v \equiv 0$ and $v \equiv 1$ give rise to the same output function; so the subsystem (4.3) is not strongly invertible. □

REMARK. This theorem seems to be an improvement of [11]. Following the notation of [11], one of the hypotheses for invertibility is that Z^j and $D\phi ad_A^{k(i)} B^i$ are linearly independent for $j = 1,\dots,q$ and $i = 1,\dots,m$ (p.208). This assumption does not imply that linear combinations of Z^j's are linearly independent of $D\phi ad_A^{k(i)} B^i$, which explicitly has been used (p.210).

COROLLARY 4.4. *The system (4.1) is strongly invertible at* x_0 *if and only if locally around* x_0 *there is no controllability distribution, which is nontrivial, contained in* Ker C_*.

PROOF. First we note that by using TSINIAS & KALOUPTSIDIS ([15]) one can easily prove (as in the linear case) that each controllability distribution arising from a k-dimensional subsystem of (4.1) also appears as controllability distribution of a single-input subsystem. Now applying theorems 4.2 and 3.2 exactly yields the result. □

Finally if we investigate strong invertibility of the system (4.1) we obtain as the analogue of theorem (3.1).

THEOREM 4.5. *The system (4.1) is strongly invertible if and only if there exists an open and dense submanifold* M_0 *of such that on* M_0 *no nontrivial controllability distribution is contained in* Ker C_*.

PROOF. Follows directly from theorem 4.3 and corollary 4.4. □

5. CONCLUSION

Necessary and sufficient conditions for strong invertibility of affine nonlinear systems are derived. The well-known condition for strong invertibility of linear multivariable systems - a linear system is strongly invertible if and only if there does not exist a controllability subspace contained in Ker C - has been generalized to nonlinear systems. From a practical point of view it would be desirable to have an algorithm for computing the largest controllability distribution in a given distribution. As already noted in [8] at the moment it is not clear if such a maximal element exists. It has been proven in [8], see also [7], that there does exist a maximal *regular* controllability distribution in a given distribution, but clearly we also have to deal with non-regular controllability distributions.

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